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Retrading in Market Games *

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Abstract

When agents are not price takers, they typically cannot obtain an efficient reallocation of resources in one round of trade. This paper presents a non-cooperative model of imperfect competition where agents can retrade allocations, consistent with Edgeworth's idea of recontracting. We show that there are allocations on the Pareto frontier that can be approximated arbitrarily closely when trade is myopic, i.e., when agents play a static Nash equilibrium at every round of retrading. We then show that the converging sequence of allocations genertated by myopic retrading can also be supported along some Subgame Perfect Equilibrium path when traders anticipate future rounds of retrading.

Keywords: Market Games, Retrading, Myopic versus Far-sighted Behavior, Pareto Optimality.

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1 Introduction

In Edgeworth (1881), we find the following definition: "A final settlement is a settlement which cannot be varied by recontract within the field of competition". In this definition of a final settlement, the emphasis is on outcomes that are immune to recontracting. When individuals interact cooperatively, outcomes immune to recontracting are defined to lie in the core of an exchange economy (Debreu and Scarf (1963)). In contrast, our emphasis is on a non-cooperative formulation of recontracting in a general equilibrium model characterized by imperfect competition. When the outcomes of trade are inefficient, traders must be allowed to reopen markets. The allocations from the previous round of trade are the initial endowments in any new round of trade, while the rules of exchange remain constant. This generates an iterative process of retrading in which traders are able to reopen markets before they consume. We focus on the issue of whether retrading will allow traders to approximate allocations on the Pareto frontier.

The non-cooperative game of exchange we use is the Shapley-Shubik market game[1] where the rules of exchange allow all traders to influence prices by sending quantity signals. With a finite number of traders, Dubey and Rogawski (1990) have shown, under some mild regularity assumptions on preferences, that the Nash equilibrium outcomes of the market game are Pareto optimal if and only if the initial endowments of traders are Pareto optimal as well[2]. This result allows us to study the incentives traders have to reopen markets before they consume their final allocations even in trading environments characterized by complete information.

In our model, traders can reopen markets a finite or infinite number of times before they consume. We think of the number of times traders can reopen markets as away of capturing the frequency with which they can retrade. At each round of trade all commodities are exchanged at trading posts except for the numeraire commodity, in which bids for all other commoditiues have to be made. For each non-numeriare commodity, traders can submit bids for the commodity and make offers

[1]Shapley and Shubik (1977), Dubey and Shubik (1978), Sahi and Yao (1989), Peck and Shell (1991).

[2]Dubey and Rogowski (1990).

of a quantity of the commodity, at the relevant trading post. In any new round of trade, the endowments of individuals are their final allocations from the previous round of trade. Using these endowments, individuals now make bids and offers in the trading posts and obtain allocations determined by the same price formation rule and allocation rule. The cost of reopening trading posts in any new round of trade is measured by a common discount factor for all traders.

We study the outcomes of myopic retrading as well as far-sighted retrading. A path of myopic retrading only requires that each period allocation be a Nash equilibrium outcome given the final allocation of the previous period. With far-sighted retrading, traders anticipate that there will be retrading in future time-periods.

With myopic retrading, we show that there are allocations on the Pareto frontier that can be approximated arbitrarily closely along some equilibrium path of retrading, as the discount factor is close enough to perfect patience and the number of allowable retrading periods is large enough. We construct an example in which there is a unique path of myopic retrading, which approximates the Pareto frontier. The same sequence of allocations that approximates a Pareto optimal allocation under myopic retrading can be sustained by a Subgame Perfect Equilibrium profile under far-sighted retrading. We are also able to establish, as a corollary, that the approximation result only requires individual traders to condition their choice of current bids and offers on the prices observed at each trading post in the preceeding round of trade and on their own final allocation from the preceeding round of trade. However, we also show that, along any equilibrium path of finite retrading, with or without far-sighted behavior, no allocation on the Pareto frontier can be attained even when the cost of reopening trading posts is negligible.

We are also able to demonstrate that any Subgame Perfect Equilibrium that sustains a sequence of allocations that converges to some allocation on the Pareto frontier must have the property that it must look increasingly similar to the sequence of allocations generated by myopic retrading. Moreover, the set of allocations supported by Subgame Perfect Equilibrium profiles is shown to expand as the cost of reopening trading posts falls. This weak monotonicity result holds with finite as well as infinite horizon.

All the results just described are first proved under the simplifying assumption that traders can consume commodities (all tradeable) only after having stopped trading. However, we show that all results extend to the more general class of games where traders can decide to consume part of their current endowment at any time, while remaining on the market with the rest.

Our model of retrading can also be derived as reduced form of a model where the tradeable goods are actually assets. The goods that agents consume can be simply viewed as derived from the flow yields of the currently owned stock of assets. With this interpretation in mind, our model of retrading can be thought of as providing a rationale for resale markets where assets (more generally, durable goods) are traded. Moreover, in this case the issue of consumption becomes irrelevant, since the assets owned by each individual at any given time cannot be consumed, they can only be kept or traded.

Finally, although we focus on the possibility of eventually reaching an efficient allocation of resources (or assets) through retrading, we point out that a new type of market failure also arises in market games with retrading: there are "bad" Subgame Perfect Equilibria where traders delay trade only because the other traders do the same.

The rest of the paper is organized as follows. The next subsection compares our retrading model and our results with the related literature. The next section presents the economy and the basic models of non-cooperative trade that we study. Section 3 gives a simple example, in which the unique equilibrium path of retrading converges to the competitive equilibrium. Section 4 characterizes the equilibria of the benchmark retrading model with myopic players. Section 5 characterizes the Subgame Perfect Equilibria of the market game with far-sighted retrading. Section 6 contains the extensions to the case where each trader can always choose between consuming and trading any subset of her own commodities and to the case where the tradeable goods are assets. Some more technical material is relegated to the appendix.

1.1 Related literature

The model we study as well as the results we obtain are different from the body of related work that studies dynamic noncooperative games of exchange.

Gale (1986a, 1986b, 1987) and McLennan and Sonnenschein (1991) look at a model where traders are repeatedly pair-wise matched and bargain over the trades that they make with each other. With a continuum of traders, complete information, and endogenous replacement, there is a stationary equilibrium which converges to the competitive equilibrium as the discount factor converges to one. A key feature of the models they study is that every trader has a apositive probability of meeting someone who is about to leave which implies that in their papers agents cannot leave immediately after having obtained a satisfactory bundle. Other differences are (1) We have a finite number of traders; (2) Gale's traders make direct transfers to each other in pairs, which are independent of the transfers made within other matched pairs at each round of trade (in contrast, in our model, trade is anonymous and each commodity is traded at a common price); (3) Once a pair agree to trade, they exit and are replaced by identical copies. In this sense, in contrast to our model, the same set of traders *never really agree to retrade* with each other along the equilibrium path of play. In that framework, retrading refers to the fact that any type has a positive probability of being repeatedly matched with any given other type of trader. Moreover, in order to obtain convergence to efficient allocations, Gale needs traders to be far-sighted. In contrast, we are able to obtain convergence when traders are myopic.

Dubey, Sahi and Shubik (1993) is closer to our paper, as they also study retrading in market games. However, they have a model with a continuum of agents[3] who, in addition, do not discount future consumption. They show that if equilibria in the one-shot market game fail to coincide with competitive equilibria due to the endowment constraints in the numeraire commodity binding for non-negligible subsets of traders, competitive equilibria can nevertheless be approximated arbitrarily

[3] A model of retrading with a continuum of agents corresponds to Edgeworth's notion of recontracting in a field of perfect competition. In contrast, our model studies recontracting in a field of imperfect competition.

when traders are allowed to reopen trading posts before they consume their final allocations. In our model, with a finite number of agents, the Nash equilibria of the market game is Pareto inefficient even when endowment constraints in the numeraire commodity don't bind for any individual trader.

The process of myopic retrading that we study in Section 4 shares with the iterative processes studied by Dreze and de la Vallee Poussin (1971), Malinvaud (1972) and Allen, Dutta and Polemarchakis (1999), the property that reallocations can be Pareto improving at each step.

Peck and Shell (1990)[4] study a model of a market game where traders can make arbitrarily large short sales, so that net trades are small relative to gross trades. Using this model they show that, at equilibrium, no individual action has a big effect on market prices, and therefore equilibrium allocations approximate competitive equilibrium allocations. Introducing the possibility of arbitrarily large short sales requires traders in their model to satisfy a budget constraint. They postulate some form of outside enforcement of the budget constraint via a bankruptcy rule. In any case, allowing for short sales has similar effects on imperfect competition as allowing for retrading (as they point out in footnote 6).

2 The Economy

We study trade in pure exchange economies with a finite set of commodities L (indexed by l), a finite set of individuals I (indexed by i). Each individual's consumption set is $\Re^L_+$, and his endowment is denoted by $w^i \in \Re^L_{++}$. The utility function is $u^i : \Re^L_+ \to \Re$. A pure exchange economy is $E = \{L, (u^i, w^i) : i \in I\}$. An allocation $x = (x^1, ..., x^I)$ such that $x^i \in \Re^L_+$ for all $i \in I$ is feasible if, in addition, $\sum\limits_{i \in I} x^i = \sum\limits_{i \in I} w^i$. A feasible allocation x is Pareto optimal if there is no other feasible allocation y such that $u^i(y^i) \geq u^i(x^i)$ for all $i \in I$ with $u^i(y^i) > u^i(x^i)$ for some $i \in I$. Throughout the paper, we keep the total endowments of each commodity fixed. Let P denote the set of Pareto optimal allocations and let IR denote the set of individually rational allocations x such that $u^i(x) \geq u^i(w)$ for all $i \in I$. Let F denote the

[4] For a related liquidity based approximation result see also Okuno and Schmeidler (1986).

set of feasible allocations, i.e., $F \equiv \{x \in \Re_+^{LI} : \sum_{i \in I} x_l^i = \sum_{i \in I} w_l^i, l = 1, ..., L\}$.

2.1 The one-shot market game

In this section we describe the Shapley-Shubik (Shapley and Shubik (1977)) market game of non-cooperative exchange[5] Each trader makes bids and offers of commodities at trading posts where commodities are exchanged; moreover, all bids are denoted in some numeraire commodity, which we set to be commodity 1. Traders are allowed to make offers in all the other commodities $2, ..., L$. There are $L-1$ trading posts for commodities $2, ..., L$. A strategic action for a trader i is a vector $s^i = (b_2^i, ..., b_L^i, q_2^i, ..., q_L^i)$ where b_l^i denotes the bid for commodity l while q_l^i denotes the offer of commodity l, $l = 2, ..., L$. The corresponding set of strategic actions for each trader i is $S^i(w^i) = \{(b_2^i, ..., b_L^i, q_2^i, ..., q_L^i)$ such that $b_l^i \geq 0, \sum_{i \in I} b_l^i \leq w_1^i$, $0 \leq q_l^i \leq w_l^i,\ l = 2, ..., L\}$. All bids and offers have to be non-negative and the offer of a commodity made by a trader cannot *exceed* his endowment of that commodity. For each profile of strategic actions $s = (s^1, ..., s^I)$, at the trading post for commodity l, the aggregate bid is $B_l = \sum_{i \in I} b_l^i$ while the aggregate offer is $Q_l = \sum_{i \in I} q_l^i$. Define the price at the trading post l to be $\pi_l(s) = \frac{B_l}{Q_l}$ if $B_l > 0$, $Q_l > 0$, with $\pi_l(s) = 0$ otherwise. For each trader i, the allocation rule determines commodity holdings as follows: If $\pi_l(s) \neq 0$, $x_1^i(s) = w_1^i - \sum_{l=2}^{L} b_l^i + \sum_{l=2}^{L} q_l^i \pi_l(s)$ and $x_l^i(s) = w_l^i - q_l^i + \frac{b_l^i}{\pi_l(s)}$, $l = 2, ..., L$. If $\pi_l = 0$, $x_l^i(s) = w_l^i$, for all $i \in I$. Let $v^i(s^i, s_{-i})$ be the payoff associated with s. A Nash equilibrium profile of strategic actions is s^* such that $v^i(s^{*i}, s_{-i}^*) \geq v^i(s^i, s_{-i}^*)$, for all $s^i \in S^i(w^i)$ and $i \in I$. A Nash equilibrium profile of strategic actions s^* such that $b_l^{i*} > 0, q_l^{i*} > 0$ for all $l = 2, ..., L$ and $i \in I$ is an interior Nash equilibrium. Let $N(w)$ denote the set of Nash equilibrium allocations of the market game.

In the one-shot market game with variable offers, observe that the trivial Nash equilibrium where $b_l^{*i} = q_l^{*i} = 0$ for all l and $i \in I$, always exists and yields the

[5]The reason for choosing this particular type of market game for our analysis is that, as will be clarified below, such a game always gives traders incentives to *retrade*. Hence, it makes sense to study the effects of retrading in this context.

initial endowments as the final allocation. When $w \in P \cap \Re^{LI}_{++}$, there is always an interior Nash equilibrium at which individuals consume their initial endowments guaranteeing that $N(w) \cap P \neq \phi$, since w would be an element of such an intersection. What happens when $w \notin P$? Consider the following three properties, which turn out to characterize the set of Nash equilibrium allocations of the one-shot market game:

- **(P1)** *(Static inefficiency)* If $w \notin P$, then $N(w) \cap P = \phi$.
- **(P2)** *(Weak gains from trade)* If $w \notin P$, there exists $x \in N(w)$ such that $u^i(x^i) \geq u^i(w^i)$ for all $i \in I$, with $u^i(x^i) > u^i(w^i)$ for some $i \in I$.
- **(P3)** *(Strong gains from trade)* If $w \notin P$, there exists $x \in N(w)$ such that $u^i(x^i) > u^i(w^i)$ for all $i \in I$.

(**P1)** requires that whenever the endowments in an exchange economy are Pareto suboptimal, there is no interior Nash equilibrium allocation that is also Pareto optimal. (**P2)** requires that whenever the endowments in an exchange economy are Pareto suboptimal, there is nevertheless some interior Nash equilibrium allocation that makes at east one trader better-off relative to his endowments. (**P3)** requires that whenever the endowments in an exchange economy are Pareto suboptimal, there is nevertheless some interior Nash equilibrium allocation that makes all traders better-off relative to their endowments.

Suppose preferences and endowments satisfy the following regularity assumption:

Assumption 1 *For each $i \in I$, u^i is strictly monotone, strictly-concave, element of C^r, $r \geq LI$, and the closure of the indifference curves through w^i are contained in $\Re^L_{++}$ and remain bounded away from the boundary of the consumption set.*

An important class of economies for which **(P1)**, (**P2), (P3)** characterize the set of static Nash equilibria whenever $w \notin P$, is identified by the following result, due to Dubey and Rogawski (1990) (see also Peck, Shell and Spear (1992) for similar results in a related market game).

Lemma 1 *(Dubey and Rogawski (1990)) (i) Suppose $w \in P$. Then, $w = N(w)$. (ii) Suppose $w \notin P$. If Assumption 1 is satisfied, then, there exists $x \in N(w)$ which satisfies* ***(P1)****,* ***(P2)****,* ***(P3)****.*

Proof. As endowments and utility functions satisfy Assumption 1, the existence of an equilibrium point, as defined and used in Theorem 1 in Dubey and Shubik (1978), also implies the existence of an interior Nash equilibrium where $b_l^{i*} > 0, q_l^{i*} > 0$ for all $l = 1, ..., L$ and $i \in I$. But then the assumptions used in Proposition 1, Remarks 1 and 5, Section 4 and Section 5.1, in Dubey and Rogowski (1990) are satisfied, and (i) and (ii) immediately follow. **QED.**

In later sections, when convenient, we state our results, we will often directly assume that one or all of **(P1)**,**(P2)**,**(P3)** characterize $N(w)$ whenever $w \notin P$, without invoking Assumption 1.

2.2 The market game with retrading

Given Lemma 1, there are gains from trade that are not exhausted after a one-period exchange. Therefore there are always *incentives to retrade.* In this section we describe an exchange mechanism that takes into account these incentives. Now, trading posts can reopen over a sequence of finite or infinite periods, $t = 0, 1, ..., T$. At each t an action for trader i is a vector s_t^i. The corresponding set of strategic actions at t for each trader i is $S_t^i(x_{t-1}^i)$, starting from $s_{-1}^i = (0, ..., 0)$ for all $i \in I$ and $x_{l,-1}^i = w_l^i$, for all $l = 1, ..., L$ and for all $i \in I$. For each strategic action profile s_t, in the trading post for commodity l, the aggregate bid is $B_{l,t}$ while the aggregate offer is $Q_{l,t}$, with the corresponding price $\pi_{l,t}(s_t)$, defined as in the static game. For each trader i, the allocations $x_t^i(s_t)$ are also defined as before. Along a sequence of action profiles $s = \{s_0, ..., s_t, ...\}$, we say that player i *stops trading* after period $\tilde{T}^i$ iff $b_{l,t'}^i = q_{l,t'}^i = 0$ for all $t' \geq \tilde{T}^i$, $l = 2, ..., L$. Even though traders can stop trading at different times, it is convenient not to complicate notation by explicitly keeping track of traders who drop out. We can do so without loss of generality as the bids and offers of a trader can be zero at any round of trade and hence a trader i who stops trading at some period $\tilde{T}^i$ can be counted as a market player who makes zero bids and offers in all periods including and subsequent to $\tilde{T}^i$.

In what follows we shall consider two models of retrading, labeled as **myopic** and **far-sighted**.

Case 1 (**Myopic retrading**): When retrading is *myopic*, at each new round of retrading traders behave in a very simple way: at each new round of retrading, they choose a vector of bids and offers that constitutes a static Nash equilibrium to the final allocation obtained from the previous round of trade. In the notation developed before, at each t, the strategy profile chosen, s_t, satisfies the condition that $x_t(s_t) \in N(x_{t-1})$. Traders consume when they stop trading. As the utility function of each trader is continuous and the set of feasible allocations compact, we remark that even when $\tilde{T}^i = \infty$, the payoff to any player i remains well-defined. We study the properties of the stationary allocations of this retrading process. Myopic traders can be seen as traders who do not expect that trading posts can be reopened, so they play their best responses as if the current trading round were the last. Consistent with this, we will study myopic retrading without discounting, even though the results extend to the case where discounting occurs.

Case 2 (**Far-sighted retrading**): When retrading is *far-sighted*, all traders know that future play will, in general, be conditioned on the outcomes of the current round of trade. Here, as before, we assume that an individual trader consumes only when she has stopped trading. However, now we endow each trader i with a common discount factor δ. When T is finite, δ lies in $[0, 1]$. When $T = \infty$, δ lies in $[0, 1)$.[6] Trader i's payoff, once she has stopped trading in period $\tilde{T}^i$, is $\delta^{\tilde{T}^i} u^i(x^i_{\tilde{T}^i})$. A history of play at period t is $h_t = \{s_0, ..., s_{t-1}\}$. The corresponding set of histories is denoted by H_t. A pure strategy for trader i is a sequence $\sigma^i = \{\sigma^i_0, ..., \sigma^i_t, ...\}$ with $\sigma^i_t : H_t \to S^i_t$ for all t. Denote by $\sigma^i|h_t$ the restriction of σ^i to the subgame from period t after history h_t. A pure strategy profile $\sigma = (\sigma^1, ..., \sigma^I)$ is a Subgame Perfect Equilibrium (SPE henceforth) if, for every h_t, the restriction $\sigma^i|h_t$ for all traders $i \in I$ is a Nash equilibrium in the subgame from period t. Let $\tilde{X}(\delta, w, T)$ denote the set of SPE allocations of the market game with far-sighted retrading, where trading posts are allowed to be reopened up to $T + 1$ times.

[6] We interpret δ as a measure of the cost of reopening trading posts in any new round of trade.

3 An example

In this section, we analyze retrading in an example. There are two individuals and two commodities. Both individuals have quasi-linear utility functions with $u^i(x) = x_1^i + f^i(x_2^i)$ (and similarly for j). We assume that $f^k(.)$ $(k = i, j)$ is strictly monotone, strictly concave, twice-continuously differentiable and satisfies the boundary condition that $\lim_{x_2 \to 0} \partial f^k(x_2) = \infty$ for $k = i, j$. Further, for simplicity, we choose the units in which commodities are measured so that $\sum_k w_2^k = 1$. We focus on retrading in the "sell-all" market game. The "sell-all" version of the Shapley-Shubik market game is obviously simpler than the variable offers version: At each time t where trader i is still active, his offer is assumed to equal $x_{2,t-1}^i$, which is the endowment of commodity 2 inherited from the trades of the previous period. Other than for this simplification, the strategies, aggregate variables, and the allocation rules are identical to the more general variable offers model described before[7]. In this case there is a unique Nash equilibrium with trade in the one-shot market game (the no trade equilibrium does not exist)[8]. This means that finitely repeated trade would not add anything, whereas we now show that finite retrading leads the traders towards the competitive allocation *even* if they are myopic.

It will be convenient to refer to $w_2^i = \alpha_0^i \in (0, 1)$ as individual i's initial share of commodity 2 and α_t^i as individual i's share at the end of round $t - 1$ of retrading. In what follows, we will make an assumption that each individual is endowed with enough of commodity 1 to ensure that the endowment constraint of the numeraire commodity does not bind. For the moment, we simply assume that at any round of trade, all traders have enough of the numeraire commodity to ensure existence of an interior one-shot Nash equilibrium in any one round of trade[9]. Using the allocation

[7]One additional difference would be in the precise definition of what it means to stop trading in the sell-all market game. We avoid the formal definitions since they are not relevant for this example, but the intuitive feature of any such definition is that traders must bid the exact amounts that give them back the endowments obtained with their last real trade.

[8]Endowments and utility functions satisfy Assumption 1, hence the existence of an equilibrium with trade follows from Dubey (1980), Remark 2. Further, using Remark 5 in Dubey and Rogowski (1990), it also follows that if $w \notin P$ then $N_f(w)$ satisfies **(P1)**, **(P2)**, **(P3)**.

[9]More formally, if retrading can take place over T periods, at any T' consider the sum

rule, we obtain that at any round of retrading t, $t = 0, 1, ...$, if the current profile of actions is $s_t = (b^j_{2,t}, b^i_{2,t})$, player $i's$ objective function at time t is

$$x^i_{1,t} - b^i_t + \alpha^i_t B_t + f^i\left(\frac{b^i_t}{B_t}\right)$$

where $B_t = b^i_{2,t} + b^j_{2,t}$. Using the fact that the ratio $\frac{b^i_t}{B_t} = \alpha^i_{t+1}$, if the current profile of actions s_t is an interior Nash equilibrium, we can rewrite the first-order conditions of traders to obtain the dynamical system that characterizes the evolution of the sequence of allocations generated by myopic retrading:

$$\frac{\partial f^i\left(\alpha^i_{t+1}\right)\left(1 - \alpha^i_{t+1}\right)}{\partial f^j\left(1 - \alpha^i_{t+1}\right)\alpha^i_{t+1}} = \frac{\left(1 - \alpha^i_t\right)}{\alpha^i_t}$$

Evidently, a stationary point of the preceeding map is an interior allocation on the Pareto frontier. Moreover, as both individuals have quasi-linear utility functions, the allocations of commodity 2 is uniquely determined at an interior Pareto optimum. Let $\bar{\alpha}^i$ denote individual $i's$ share of commodity 2 at the interior Pareto optimum. Suppose $\alpha^i_0 < \bar{\alpha}^i$. Then, as $f^k(.)$ is strictly concave, we must have that $\frac{\partial f^i\left(\alpha^i_0\right)}{\partial f^j\left(1-\alpha^i_0\right)} > 1$. Moreover, $\frac{\left(1-\alpha^i_0\right)}{\alpha^i_0} > \frac{\left(1-\bar{\alpha}^i\right)}{\bar{\alpha}^i}$. For all $t > 0$ such that $\frac{\partial f^i\left(\alpha^i_t\right)}{\partial f^j\left(1-\alpha^i_t\right)} > 1$, $\frac{\left(1-\bar{\alpha}^i\right)}{\bar{\alpha}^i} < \frac{\left(1-\alpha^i_t\right)}{\alpha^i_t} < \frac{\left(1-\alpha^i_{t-1}\right)}{\alpha^i_{t-1}}$. If there exists $\hat{t}$ such that $\frac{\partial f^i\left(\alpha^i_{\hat{t}}\right)}{\partial f^j\left(1-\alpha^i_{\hat{t}}\right)} < 1$, $\frac{\left(1-\bar{\alpha}^i\right)}{\bar{\alpha}^i} > \frac{\left(1-\alpha^i_{\hat{t}}\right)}{\alpha^i_{\hat{t}}}$ and as long as for all $t > \hat{t}$, $\frac{\partial f^i\left(\alpha^i_{\hat{t}}\right)}{\partial f^j\left(1-\alpha^i_{\hat{t}}\right)} < 1$, we must have that $\frac{\left(1-\bar{\alpha}^i\right)}{\bar{\alpha}^i} > \frac{\left(1-\alpha^i_t\right)}{\alpha^i_t} > \frac{\left(1-\alpha^i_{t-1}\right)}{\alpha^i_{t-1}}$.

$\sum_{t=0}^{T'-1}\left(\frac{1-\alpha^i_{t+1}\left(1-\alpha^i_{t+1}\right)}{\alpha^i_t}\right)\frac{\alpha^i_t}{1-\alpha^i_t}\partial f^i\left(\alpha^i_{t+1}\right)$. As $\lim_{x_2\to 0}\partial f^i(x_2) = \infty$ and $w^i_2 > 0$ for $i \in I$, without loss of generality we can assume that α_t is an element of some compact set bounded away from zero and one, independent of t (equivalently, $b^i_t \geq \xi^i > 0$, for all i). Let $K^{i,T'}$ denote the maximum of the above sum taken over α_t, $t < T'$, which lie in this compact set. We assume that $w^k_1 - \max_{T \geq T' \geq 1} K^{k,T'} > 0$. This ensures that in any round of retrading each individual i has enough of the numeraire commodity to ensure that an interior solution to the maximization problem $Max_{0\leq b^i_t \leq x^i_{1,t-1}} x^i_{1,t} - b^i_t + \alpha^i_t B_t + f^i\left(\frac{b^i_t}{B_t}\right)$ i.e. the endowment constraint of the numeraire commodity never binds as we must have that for all $b^j_t \geq \xi^j > 0$, $\frac{\partial f^i\left(\alpha^i_{t+1}\right)\left(1-\alpha^i_{t+1}\right)}{\partial f^j\left(1-\alpha^i_{t+1}\right)\alpha^i_{t+1}}$ evaluated at $b^i_t = x^i_{1,t-1}$ is strictly less than $\frac{\left(1-\alpha^i_t\right)}{\alpha^i_t}$.

Suppose there exists $\tilde{t} > \hat{t}$ such that $\frac{\partial f^i\left(\alpha^i_{\tilde{t}}\right)}{\partial f^j\left(1-\alpha^i_{\tilde{t}}\right)} > 1$. Consider the ratio

$$\frac{\partial f^i\left(\alpha^i_{\tilde{t}}\right)...\partial f^i\left(\alpha^i_{\hat{t}}\right)}{\partial f^j\left(1-\alpha^i_{\tilde{t}}\right)...\partial f^j\left(1-\alpha^i_{\hat{t}}\right)}.$$

Notice that if the above ratio is equal to one we must be on the Pareto frontier. On the other hand the above ratio must be strictly greater than one as otherwise $\frac{1-\alpha^i_{\tilde{t}}}{\alpha^i_{\tilde{t}}} > \frac{\left(1-\bar{\alpha}^i\right)}{\bar{\alpha}^i}$ a contradiction. Therefore, $\frac{\partial f^i\left(\alpha^i_{\tilde{t}}\right)...\partial f^i\left(\alpha^i_{\hat{t}}\right)}{\partial f^j\left(1-\alpha^i_{\tilde{t}}\right)...\partial f^j\left(1-\alpha^i_{\hat{t}}\right)} > 1$, which implies that $\frac{\left(1-\bar{\alpha}^i\right)}{\bar{\alpha}^i} < \frac{1-\alpha^i_{\tilde{t}}}{\alpha^i_{\tilde{t}}} < \frac{\left(1-\alpha^i_{\tilde{t}-1}\right)}{\alpha^i_{\tilde{t}-1}}$. By repeating the above argument from $\tilde{t}$, it follows that the ratio $\frac{\left(1-\alpha^i_t\right)}{\alpha^i_t}$ converges to $\frac{\left(1-\bar{\alpha}^i\right)}{\bar{\alpha}^i}$ and therefore, α^i_t to $\bar{\alpha}^i$. A symmetric argument establishes convergence when $\alpha^i_0 > \bar{\alpha}^i$. An immediate consequence is that the sequence of allocations generated by myopic retrading must converge to the Pareto frontier. Moreover, note that from the equations that determine final allocations, we also obtain that individuals consumption of commodity 1 is identical to that at the competitive equilibrium.

What about far-sighted retrading? We show that the sequence of allocations generated by myopic retrading can be supported as SPE outcomes when T is very large but finite. Consider the sequence of allocations $y_1, ..., y_t, ...$, with $y_0 = w$, associated with myopic retrading. Note that $y_t = N_f(y_{t-1})$,[10] $t = 1, ...$, with the associated sequence of payoffs $u(y_1), ..., u(y_t), ...$ in utility space $\Re^2$. Consider the following strategy profile $\tilde{\sigma}$. For $t \leq \underline{T} + 1$, play $\tilde{s}_t$ such that $y^i_t = x^i(\tilde{s}_t)$ (and $\tilde{u}^i_t = u^i(x^i(\tilde{s}_t)))$ as long as $\tilde{h}_t = \{\tilde{s}_0, ..., \tilde{s}_{t-1}\}$; otherwise, if there has been a deviation, play s'_t such that $x(s'_t) = N_f(x_{t-1})$, for all $t \leq T$. We need to show that $\tilde{\sigma}$ is a SPE. By construction, after any deviation, both players continue to choose bids according to one-shot Nash equilibria. As all sequences of allocations generated by one-shot Nash equilibria converge to the same allocation for both commodities, no player has an incentive to deviate when T is large.

Finally, it is worth noting that this example generalizes to more goods and individuals with quasi-linear utility functions.

[10] The subscript f refers to "fixed" offers, since in this example offers are not strategic.

4 Myopic retrading

We start the general analysis with myopic retrading. We show that, starting from an arbitrary configuration of initial endowments, traders are able to converge to some allocation in the Pareto set. Nevertheless, convergence cannot take place in a finite number of rounds of myopic retrading. We state the results only for the market game with variable offers described in Section 2.2, but we note that the same results also obtain for the "sell-all" market game (as one could guess from the previous section).

The following definition identifies the sequences of allocations that are consistent with myopic retrading.

Definition of Myopic Retrading: A sequence of allocations $\{x_t\}$, $t = 1, ...$ is generated by myopic retrading if and only if it satisfies the inclusion $x_t \in N(x_{t-1})$, for all $t \geq 1$[11]. An allocation y is stationary with myopic retrading if and only if $y = N(y)$.

Some notation is needed before proving the main result of this section. For any allocation y, let $u(y) = (u^1(y^1), ..., u^I(y^I))$. For any $K \subset \Re^{LI}$, let $u(K) = \{u(y) : y \in K\}$. Observe that $u(K) \subset \Re^I$, for all K. Let $\|.\|$ denote the Euclidian norm. Then, we define the distance between a vector y and a set K as $d(u(y), u(K)) \equiv \inf_{\hat{u} \in u(K)} \|u(y) - \hat{u}\|$.

Proposition 1 *Consider $w \in \Re^{LI}_{++}$, suppose that $N(w)$ satisfies **(P1)-(P2)** whenever $w \notin P$. Then, for any $w = y_0 \in \Re^{LI}_{++}$, there exists a sequence of allocations $\{\tilde{y}_t\}$, $t = 0, 1, ...$, $\tilde{y}_t \in N(\tilde{y}_{t-1})$ for all $t \geq 1$, such that, for any $\varepsilon > 0$, there is a $T > 0$ with $d(u(\tilde{y}_t), u(P \cap IR)) < \varepsilon$ for all $t > T$.*

Proof. If $w \in P$, then $w = N(w)$ and we are done. Therefore assume that $w \notin P$. Consider the sequence of sets $N_1, ..., N_t, ...$, with $y_0 = w$, and $N_t = \{x : x \in N(y),\ for\ some\ y \in N_{t-1}\}$, $t = 1, ...$, with the associated sequence of sets $u(N_1), ..., u(N_t), ...$ in utility space $\Re^I$. By **(P2)**, we can extract a sequence $\tilde{u}_t$, $t = 0, 1, ...$ such that $\tilde{u}_t \in u(N_t)$ and $\tilde{u}_{t+1} > \tilde{u}_t$, at each t, with $y_0 = w, ..., y_t, ...$

[11] x_0 is obviously the initial endowment w.

the associated sequence of allocations. Note that for each $i \in I$, the sequence $\tilde{u}_t^i$, $t = 0, 1, ...$ is bounded above, as the utility of each individual is continuous and the set of feasible allocations is compact. Let $\bar{u}^i$ denote the supremum of the sequence $\tilde{u}_t^i$, $t = 0, 1, ...$ As every increasing sequence converges to the supremum, it follows that the sequence $\tilde{u}_t$, $t = 0, 1, ...,$ converges to $\bar{u} = (\bar{u}^1, ..., \bar{u}^I)$, the component-wise supremum of $\tilde{u}_t = (\tilde{u}_t^1, ..., \tilde{u}_t^I)$, $t = 0, 1, ...$ Moreover, it also follows that the associated sequence of allocations $y_t, t = 0, 1, ...$ converges to some allocation $\bar{y}$ such that $u(\bar{y}) = \bar{u}$. By considering every sequence of utilities and the corresponding sequence of allocations generated by myopic retrading which satisfy (**P2**), we obtain a set of allocations $\bar{Y}_w$ which consists of the limit allocations of each sequence of allocations $y_t, t = 0, 1, ...$ If we show that there exists some sequence of allocations $y_0, ..., y_t, ...$generated by myopic reatrding which satisfies **(P2)** and converges to an allocation $\bar{y} \in \bar{Y}_w$ such that $\bar{y}$ is stationary under myopic retrading, we are done: in fact, by Lemma 1, if $\bar{y} = N(\bar{y})$, $\bar{y} \in P$. To this end, define the binary relation $\prec_w$ on F (the set of feasible allocations) so that given two feasible allocations x and y, $y \prec x$ if $x \in N_t$ and $y \in N_{t'}$ for some N_t, $N_{t'}$ in the sequence of sets $N_1, ..., N_t, ...,$ with $y_0 = w$, and $N_t = \{x : x \in N(y),\ for\ some\ y \in N_{t-1}\}$, $t = 1, ...$ and either (a) $x \in N(y)$ and $u^i(x) \geq u^i(y)$ for all $i \in I$ and $u^i(x) > u^i(y)$ for some $i \in I$ or (b) x is the limit of a sequence of allocations $\{x_t\}$, $t = 1, ...$ with $x_t \in N(x_{t-1})$, for all $t \geq 1$ where $x_0 = y$. Remark that $\prec_w$ is transitive: if $y \prec_w x$ and $x \prec_w z$, then by (b) $y \prec_w z$. Therefore, $\prec_w$ is a partial order on F (page 13, Kelley (1955)). Consider any sequence of allocations $y_0, ..., y_t, ...$ generated by myopic retrading satisfying **(P2)**. Note that either $y_t \prec_w y_{t'}$ or $y_{t'} \prec_w y_t$ for all $t \neq t'$ and moreover, if $y_t \prec_w y_{t'}$ and $y_{t'} \prec_w y_t$, $y_t = y_{t'}$. Therefore, $\prec_w$ is a linear ordering over any sequence of allocations generated by myopic retrading satisfying **(P2)** (page 14, Kelley (1955)) and therefore, any sequence $y_0, ..., y_t, ...$ is a chain given the binary relation $\prec_w$ (page 15, Kelley (1955)). By Kuratowski's lemma (page 33, Kelley (1955)), each chain in a partially ordered set is contained in a maximal chain. Moreover, any chain in F, under the binary relation $\prec_w$, is a subset of some sequence of allocations generated by myopic retrading satisfying **(P2)**. Hence, the set of sequences generated by myopic retrading satisfying **P2** contains, in particular, a maximal chain ordered according to $\prec_w$. We have already shown that every sequence of allocations $y_0, ..., y_t, ...$ generated by myopic retrading satisfying **(P2)** converges to some allocation $\bar{y}$. $y_t \prec_w \bar{y}$ for all t and therefore, $\bar{y}$ is an upperbound (page 13, Kelley (1955)) of the chain consisting of the sequence of allocations $y_0, ..., y_t, ...$ under the binary relation $\prec_w$. By Zorn's lemma (page 33, Kelley (1955)), there is a maximal element of the set F under the binary relation $\prec_w$. By definition, a maximal element cannot preceed any other element of F. It follows that there is some sequence of allocations generated by myopic retrading satisfying **(P2)** that contains a maximal point. From this, it follows that, as the maximal element of a sequence of allocations generated by myopic retrading satisfying **(P2)** must be its upper bound and therefore, its limit allocation, there is some $\bar{y} \in \bar{Y}_w$ such that $\bar{y} = N(\bar{y})$ and therefore, $\bar{y} \in P$. **QED.**

The above proposition demonstrates that starting from an arbitrary Pareto suboptimal vector of initial endowments, there is some sequence of allocations, generated by myopic retrading, that converges to an allocation that is stationary under myopic retrading and therefore, to some allocation on the Pareto set[12]. Note that each profile of actions along the myopic retrading sequence constitutes a static Nash equilibrium to the allocation inherited from the preceeding round of trade. By **(P2),** for every configuration of Pareto suboptimal endowments, there is a static Nash equilibrium at which allocations are such that every trader is at least as well-off and some trader(s) strictly better-off relative to their initial endowments. This implies that the sequence of utility profiles associated with the sequence of allocations generated by myopic retrading is an increasing sequence. But then, along each dimension, corresponding to a specific individual, this sequence of utilities must converge to its supremum, which in turn determines the limit of the sequence of utility profiles generated by myopic retrading. Using the continuity of utility functions we can conclude that the limit of the sequence of allocations generated by myopic retrading yields the utility vector that is the limit of the sequence of utilities generated by myopic retrading. Consider the set of all limit allocations generated by such

[12]See the appendix for an alternative proof based on a different idea.

sequences. We show that there will always be some limit allocation in this set that is stationary under myopic retrading and therefore, on the Pareto frontier. In order to demonstrate this, we define a binary relation on the set of feasible allocations. This binary relation is a suborder of strict pareto dominance. We show that each sequence of allocations generated by myopic retrading satisfying **(P2)** is a linearly ordered chain under this binary relation. In fact, under this binary relation, any linearly ordered chain is a subset of some sequence of allocations in the set of all sequences of allocations generated by myopic retrading satisfying **(P2).** By Kuratowski's lemma, this set must, therefore, contain a maximally ordered chain under this binary relation. As each sequence of allocations generated by myopic retrading has an upper bound (as the set of feasible allocations is compact), by Zorn's lemma, some sequence of allocations in the set of all sequences of allocations generated by myopic retrading satisfying **(P2)**, has a maximal element. If a sequence of allocations generated by myopic retrading satisfying **(P2)** has a maximal element, then the maximal element must be its upper bound and therefore, its limit allocation. But, then, this limit allocation is stationary under myopic retrading and therefore, a Pareto optimal allocation[13].

Evidently, the preceeding proposition goes through with the stronger requirement that $N(w)$ satisfies **(P3)** whenever $w \notin P$. The result (as well as the next one) can also be extended to $\delta < 1$. The reason for dealing only with $\delta = 1$ in the propositions of this section is that this is the case where the other assumptions of myopic retrading make the most sense intuitively: in fact, a myopic player who does not discount the future can be assumed to believe he will consume right away. So myopia here means that traders can't predict that after trading they will change their mind, trading again instead of consuming.

Remark 1 At each stage of myopic retrading, the final allocation from the preceeding round of trade defines the distribution of endowments for a "new" economy. As the sequence of allocations converge to some allocation on the Pareto frontier, in

[13]For a similar argument see Allen, Dutta and Polemarchakis (1999) (see observation 8 page 16) who study a convergent iterative process where at each step individuals exchange assets to share risks inherent in the multiplicity of competitive equilibria.

limit, we obtain an economy with Pareto optimal endowments. As no trade is the only outcome at the competitive equilibrium of an economy with Pareto optimal endowments, in this sense the converging sequence of allocations associated with myopic retrading converges to competitive equilibria of the limit economy as well.

Although Proposition 1 demonstrates that traders will obtain allocations in the vicinity of the Pareto set, it still leaves open the question of whether traders are able to converge to an allocation *on* the Pareto frontier after a *finite* number of rounds of myopic retrading.

Proposition 2 *If $w \notin P$ and $N(w)$ satisfies* ***(P1)****, there is no $T < \infty$, and no sequence of allocations $\{y_t\}$, $t = 1, ...,$ $y_t \in N(y_{t-1})$, with $y_0 = w$ and $t = 0, ..., T$, such that $y_T \in P$.*

Proof. Given that $y_T \in P$, $y_T = N(y_T)$. Moreover, as $w \notin P$, there must be some $T' < T$ such that the allocation obtained at $T' - 1$, $y_{T'-1}$, is not in P, while for all $t \geq T'$, $y_t \in P$. Then we must have that $y_{T'} \in N(y_{T'-1}) \cap P$: a contradiction with Lemma 1. **QED.**

The intuition behind this result is simple. If trade concludes after some finite length of time, at some finite stage in the game it must be the case that while the traders' inherited allocation from the previous period is Pareto suboptimal, the final allocation they obtain after reopening trading posts is both (a) a Pareto optimal allocation, and (b) satisfies the inequalities for a Nash equilibrium allocation for the one-shot market game with the traders' inherited allocation as the endowment. But by **(P1),** with Pareto suboptimal endowments, no Nash equilibrium allocation of the one-shot market game can ever be Pareto optimal. This guarantees that no allocation *on* the Pareto set will be attained by traders after a *finite* number of rounds of myopic retrading. Without discounting, this implies that trading posts will *always* be reopened. This makes the assumption of myopic traders hard to swallow, but the next section shows that not only the results above extend to far-sighted behavior, but also that far-sighted behavior becomes indistinguishable from myopic behavior over the process of retrading.

5 Far-sighted retrading

Let us now allow traders to be far-sighted. The approximation result of Proposition 1 is confirmed, and we show that the set of Subgame Perfect Equilibrium (SPE) paths "converge" to the set of myopic retrading paths as traders keep retrading. Far-sighted retrading can lead the economy to Pareto improvements "faster" than myopic retrading, hence one would be tempted to conjecture that the more traders are far-sighted the more useful retrading is from an efficiency standpoint. However, we also show that a new kind of market failure emerges with far-sighted retrading: traders may delay trade along a SPE path merely because they expect other traders to do the same. Finally, we show that as traders become more patient, the set of SPE allocations expands.

Note first that there always is a SPE where $b^i_{l,t} = q^i_{l,t} = 0$ for all $t = 0, 1, ...,$ $l = 2, ..., L$, and $i \in I$.[14] Note also that the assumption that each trader consumes when he stops trading is only made for simplicity, and it is not crucial for the results, as discussed in Section 6 below.

The next proposition and its corollary provide a negative result, which strengthen the result of Proposition 2.

Proposition 3 *If $w \notin P$, and $N(w)$ satisfies **(P1)**, $\tilde{X}(\delta, w, T) \cap P = \phi$, for all $\delta \in [0, 1]$, and all $T < \infty$.*

Proof. Let $\bar{T} \leq T$ be the first period at which an allocation $x_{\bar{T}} \in P$ is obtained along some SPE path. Given that trade cannot take place after reaching the Pareto set, it must be the case that traders stop trading at some $\bar{T} \leq T$, i.e., $b^i_{l,t} = q^i_{l,t} = 0\ \forall i, \forall l, \forall t > \bar{T}$. Moreover, since $\bar{T}$ is the first period where p is reached, $x_{\bar{T}-1} \notin P$. As $x_{\bar{T}-1} \notin P$, by **(P1),** $N(x_{\bar{T}-1}) \cap P = \phi$. This is a contradiction, since, at the last round of trade, any SPE profile requires the final allocation to be in the set of Nash equilibrium allocations with respect to the inherited allocation. **QED.**

[14]In the sell-all version analyzed in the example the unique equilibrium has trade, but with variable offers the no trade equilibrium is always a possibility.

Corollary 1 *If $w \notin P$, and $N(w)$ satisfies* ***(P1)****, $\tilde{X}(\delta, w, \infty) \cap P = \phi$, for all $\delta \in [0,1)$.*

Proof. When $\delta \in [0,1)$, any trader gets a payoff of zero if he trades indefinitely. Therefore, along any SPE path, all traders will stop trading after some finite length of time, implying that there exists a $\bar{T} < \infty$ such that $b^i_{l,t} = q^i_{l,t} = 0$ for all $t \geq \bar{T}$, $l = 2, ..., L$. Trade stops before $\bar{T}' = \inf_T \{T : b^i_{l,t} = q^i_{l,t} = 0 \ for \ all \ t \geq T, \ l = 2, ..., L, \ i \in I\}$. Then, the proof immediately follows from Proposition 3. **QED.**

Even far-sighted traders cannot obtain allocations on the Pareto set. As trade always concludes after some finite length of time, at some finite stage in the game, it must be the case that both the traders' inherited allocation from the previous period and the final one are Pareto suboptimal, otherwise there would be a contradiction with **(P1)**. We now extend the approximation result obtained under myopic retrading to this world of far-sighted players.

Proposition 4 *If $N(w)$ satisfies* ***(P1)-(P3)*** *whenever $w \notin P \cap \Re^{LI}_{++}$, then, for every $\varepsilon > 0$, there is a $\underline{T}$ and $\underline{\delta}$ and $y \in \tilde{X}(\delta, w, T)$ such that $d(u(y), u(P)) < \varepsilon$ for all $\delta \in [\underline{\delta}, 1]$, $T \geq \underline{T}$.*

Proof. Using Proposition 1, for δ close to 1 we obtain that whenever $w \notin P \cap \Re^{LI}_{++}$, if $N(w)$ satisfies **(P1)-(P3)** whenever $w \notin P$, for any $w = y_0 \in \Re^{LI}_{++}$, there exists a sequence of allocations $\{\tilde{y}_t\}$, $t = 0, 1, ...,$ $\tilde{y}_t \in N(\tilde{y}_{t-1})$ for all $t \geq 1$, such that, for any $\varepsilon > 0$, there is a $T > 0$ with $d(u(\tilde{y}_t), u(P \cap IR)) < \varepsilon$ for all $t > T$. Now construct the following strategy profile $\tilde{\sigma}$. For $t \leq \underline{T} + 1$, play $\tilde{s}_t$ such that $y^i_t = x^i(\tilde{s}_t)$ (and $\tilde{u}^i_t = u^i(x^i(\tilde{s}_t))$) as long as $\tilde{h}_t = \{\tilde{s}_0, ..., \tilde{s}_{t-1}\}$; otherwise, if there has been a deviation, play $b^i_{\bar{t}} = q^i_{\bar{t}} = 0$, $i \in I$, for all $\bar{t} > t$. Finally, when $t > \underline{T} + 1$, play $b^i_t = q^i_t = 0$. To complete the proof, we need to show that $\tilde{\sigma}$ is a SPE. By construction, observe that no player has an incentive to deviate after $\underline{T} + 1$ or in any subgame following a deviation from the SPE path. It remains to check that no player has an incentive to deviate at any $t \leq \underline{T} + 1$. Indeed, consider player i who deviates at t choosing some action s'^i_t. As $b^i_{t'} = q^i_{t'} = 0$, $i \in I$, for all $t' > t$, denote i's maximum payoff from such a deviation by $\delta^{t+1} v^i(s'^i_t, \tilde{s}_{-i,t})$, where $x^i(s'^i_t, \tilde{s}_{-i,t})$ is the resulting allocation for i

when i chooses $s_t'^i$ while all other players choose according to $\tilde{\sigma}$. On the other hand, his payoff from continuing to choose according to $\tilde{\sigma}$ is $\delta^{T^i(\tilde{\sigma})}u^i(y)$. As $y_t \in N(y_{t-1})$, we must have

$$v^i(s_t'^i, \tilde{s}_{-i,t}) \leq u^i(y_t^i) < u^i(y^i_{T^i(\tilde{\sigma})})$$

Consider $\delta^{T^i(\tilde{\sigma})}u^i(y^i_{T^i(\tilde{\sigma})}) - \delta^{t+1}v^i(s_t'^i, \tilde{s}_{-i,t}) = \delta^{t+1}[\delta^{T^i(\tilde{\sigma})-t-1}u^i(y^i_{T^i(\tilde{\sigma})}) - v^i(s_t'^i, \tilde{s}_{-i,t})]$. Let δ_i^{t+1} be such that $[\delta^{T^i(\tilde{\sigma})-t-1}u^i(y^i_{T^i(\tilde{\sigma})}) - v^i(s_t'^i, \tilde{s}_{-i,t})] = 0$. Set $\underline{\delta} = \inf_{i,0\leq t\leq T^i(\tilde{\sigma})-1} \delta_i^{t+1}$. It follows that for all $\delta \in [\underline{\delta}, 1)$, $\tilde{\sigma}$ is a SPE strategy profile. **QED.**

The proof of Proposition 4 selects a sequence of allocations generated by myopic retrading that converges to the Pareto set and shows that such a sequence of allocations can be supported as SPE outcome with far-sighted retrading when traders are sufficiently patient. In order to explain the main idea of the proof, it is convenient to restrict attention to the case where $\delta = 1$. The strategy profile is constructed so that traders continue to choose the bids and offers that implement the sequence of allocations generated by myopic trade. If a trader deviates at some round of trade, in all subsequent rounds of trade all traders make null bids and offers at the trading post for each commodity, thus ensuring that no trade is the outcome. In the no trade phase, no individual trader has an incentive to deviate. As no other trader is making positive bids and offers on any trading post, whatever an individual trader does makes no difference to her final allocation. Given this, no individual trader has an incentive to deviate from the sequence of bids and offers that implement the sequence of allocations generated by myopic retrading. This is because the bids and offers at any round of trade constitute a static Nash equilibrium to the final allocation from the previous round of trade, which implies no individual trader can gain by deviating, as a deviation will be followed by no trade in all subsequent rounds. Under **(P3)**, all traders *strictly* gain in utility along some sequence of allocations generated by myopic retrading. This implies that if traders are sufficiently patient, they will prefer to retrade over consuming their current allocation.

Remark that in order to obtain the above approximation result, strategies *need not* to be conditioned on the observability of individual deviations. As stated, Proposition 4 implicitly assumes that each trader chooses strategies in the market game with retrading that are conditioned on the entire history of play. However, the following corollary shows that Proposition 4 goes through even when strategies are conditioned only on a subset of the entire history of play. Denote by σ_M an *anonymous* strategy profile where each player i conditions his choice of bids and offers in period t, (b_t^i, q_t^i), only on the preceeding period's *aggregate* bids and offers, $B_{l,t-1}$ and $Q_{l,t-1}$, $l = 2, ..., L$ (and therefore on the preceeding period's market price vector $\pi_{t-1}(s_{t-1})$), and on her own individual allocation $x_{t-1}^i(s_{t-1})$. Let $\tilde{X}_M(\delta, w, T)$ the set of SPE allocations for strategy profiles in Σ_M.

Corollary 2 *For every $\varepsilon > 0$, there is a $\underline{T}$ and $\underline{\delta}$ and $y \in \tilde{X}_M(\delta, w, T)$ such that $d(u(y), u(P \cap IR)) < \varepsilon$ for all $\delta \in [\underline{\delta}, 1]$, $T \geq \underline{T}$.*

Proof. It is sufficient to observe that the sequence of allocations along the SPE path, $y_0, ..., y_t, ...$, used in the proof of Proposition 4 can also be supported by a strategy profile $\tilde{\sigma}_M$ specified as follows. For $t \leq \underline{T}$, play $\tilde{s}_t$ as long as $B_{t-1} = \tilde{B}_{t-1}$ and $Q_{t-1} = \tilde{Q}_{t-1}$; otherwise, if there has been a deviation, play $b_{\bar{t}}^i = q_{\bar{t}}^i = 0$, $i \in I$, for all $\bar{t} > t$. Finally, when $t > \underline{T}$, play $b_t^i = q_t^i = 0$. It is immediate that $\sigma_M \in \Sigma_M$ is also a SPE strategy profile. **QED.**

Remark 2 The requirement that agents use anonymous strategy profiles justifies the construction of the SPE profile of strategies in the proof of Proposition 4. In that proof, any deviation off the equilibrium path of play is punished by all traders choosing no trade. Suppose all traders are sufficiently patient and that there are at least three active traders on each side of a trading post. Consider a SPE strategy profile where deviations off the equilibrium path of play are not punished by no trade. What prevents a trader from deviating from the equilibrium path of play profile of bids and offers? By definition, the current profile of bids and offers is a one-shot Nash equilibrium. It follows that a trader will deviate if she anticipates that in the continuation subgame, there is a equilibrium which supports an allocation where she is better off relative to the allocation obtained as the limit of the sequence of allocations along the SPE path of play. To prevent such a contingency from occuring, the other traders will have to coordinate on a strategy profile which is (a) an equilibrium in the continuation subgame and (b) punishes the deviating trader. Even if such a strategy profile exists, in general, following a deviation, bids and offers in the continuation subgame will be conditioned on the identity of the deviator. However, if the only observable history is that of past aggregate variables, with at least three active traders on each side of a trading post, no deviation from an equilibrium profile can ever be attributed to a specific individual. More generally, when keeping track of individual deviations may not be viable, agents will use anonymous strategy profiles, which in turn will require, in general, that deviations off the equilibrium path of play will be punished by all traders choosing no trade. This highlights the minimal nature of the information used in obtaining the approximation result in Proposition 4.

Are there other SPE strategy profiles, which do not require players to implement allocations generated by myopic retrading but which, nevertheless, supports a sequence of allocations that approximate the Pareto frontier? While the answer is generally yes, the next proposition shows that any SPE strategy profile must, after some length of time, begin to look like a strategy profile that implements allocations generated by myopic retrading. Formally, for $T = \infty$, for any SPE strategy profile σ, let $y_1(\sigma), ..., y_t(\sigma), ..., y_{T_\sigma}(\sigma)$ (where $y_t = x(s_t(\sigma))$) denote the allocations generated along the equilibrium path of play associated with σ and T_σ denotes the last period *with trade* under σ. For $\tilde{T} < T_\sigma$, let $y_1(\sigma), ..., y_t(\sigma), ..., y_{\tilde{T}}(\sigma)$ denote a $\tilde{T}$ truncation of T_σ. We say that a SPE strategy profile σ approximates the Pareto frontier if for $\varepsilon > 0$ there exists $\tilde{T} \leq T_\sigma$ and $y_{\tilde{T}}(\sigma)$ such that $d(u(y_{\tilde{T}}(\sigma)), u(P)) < \varepsilon$. Moreover, for any $\epsilon > 0$, and $w \in R^{LI}_{++}$, let $N_\epsilon(w)$ denote the set of non-trivial $\epsilon-$Nash equilibrium allocations.[15]

Proposition 5 *For any SPE strategy profile σ that approximates the Pareto frontier, for every ϵ, there is a $\tilde{T} < T_\sigma - 1$ such that for each $t > \tilde{T}$, $y_t(\sigma) \in N_{\epsilon,t}(y_{t-1}(\sigma))$.*

Proof. Consider the sequence of strategies along the equilibrium path of play of σ, $s_1(\sigma), ..., s_t(\sigma), ...s_{T_\sigma}(\sigma)$. At any t such that $y_t(\sigma) \notin N_\epsilon(y_{t-1}(\sigma))$, there is

[15] A non-trivial $\epsilon-$Nash equilibrium allocation x satisfies the condition that there is a profile s' with $x^i = x^i(s')$ such that for all $i \in I$, $u^i(x^i(s')) \geq u^i(x^i(s^i, s'_{-i})) - \epsilon$ for all $s^i \in S^i(w)$.

some player i whose maximum payoff from a deviation, denoted by $v^i(s_t^i, s_{-i,t}(\sigma))$, where $x^i(s_t^i, \tilde{s}_{-i,t}(\sigma))$ is the resulting allocation for i when she chooses s_t^i while all other players choose according to σ, is such that $v^i(s_t^i, s_{-i,t}(\sigma)) - u^i(y_t^i(\sigma)) > 0$. By choosing $b_{t'}^i = q_{t'}^i = 0$, for all $t' > t$, player i can obtain a payoff $\delta^{t+1} v^i(s_t^i, s_{-i,t}(\sigma))$. As σ is SPE, it follows that $\delta^{t+1} v^i(s_t^i, s_{-i,t}(\sigma)) \leq \delta^{\tilde{T}+1} u^i(y_{t'}^i(\sigma))$ for all $\delta \in [\hat{\delta}, 1]$ and $t' > t$ and therefore, $u^i(y_{t'}^i(\sigma)) > v^i(s_t^i, s_{-i,t}(\sigma)) > u^i(y_t^i(\sigma))$. As σ approximates the Pareto frontier, for every $\epsilon > 0$, there exists $\tilde{T}$ such that if $t > \tilde{T}$ and $t' > t$, $u^i(y_{t'}^i(\sigma)) - u^i(y_t^i(\sigma)) < \epsilon$ and therefore, $v^i(s_t^i, s_{-i,t}(\sigma)) - u^i(y_t^i(\sigma)) < \epsilon$ which implies that $y_{t+1}(\sigma) \in N_{\epsilon_t}(y_t(\sigma))$. **QED.**

When at some t players do not choose bids and offers according to myopic retrading, along they obtain an allocation $y_t \notin N(y_{t-1})$ (where y_{t-1} is the allocation obtained from $t-1$). This implies that there must be some individual i who would have incentive to deviate from the SPE strategy profile at t and then choose $b_{t'}^i = q_{t'}^i = 0$ for all $t' > t$. Therefore, if $\{y_t : t \geq 0\}$ is generated along some SPE path of play, it must be the case that the gain in utility for i in the continuation game along the SPE path of play from $t+1$ outweighs the gain in utility from deviating at t. As we approach the Pareto frontier along a SPE, an individual's gain in the continuation game along the SPE path becomes smaller, and so must the gain in utility by deviating from the equilibrium path of play. Remark that a similar result goes through for when T is large but finite (simply substitute T for T_σ throughout).

It is intuitive that far-sighted retrading can lead the economy within a given neighborhood of the Pareto frontier faster than myopic retrading. To see this, consider $\delta < 1$ and T such that $d(u(y_T), U(P \cap IR)) < \epsilon$, where y_T is the allocation obtained through the myopic retrading path $y_0, ..., y_t, ..., y_T$. The same path can be sustained as a SPE path of far-sighted retrading, hence far-sighted traders can always do at least as well as myopic traders. They can also do strictly better: Suppose that $T > 3$; then it is possible to construct a SPE profile where at the first round of trade the obtained allocation is directly y_{T-1} as long as y_{T-1} can be attained by some combination of bids and offers with the initial endowments w. The threat of no trade in the last round can make deviations from this profile not attractive, if δ is high enough and the game satisfies **P3**. Does this mean that far-sighted retrading *always* leads to greater gains in efficiency? The answer is no, and the following remark shows that there are also SPE of the game that make traders worse off than with just one round of trade.[16] A *new* type of *market failure* can also arise: there are SPE where traders will delay trade only because all other traders do the same.

Remark 3 By Lemma 1, we know that there always exists a static Nash equilibrium in the one-shot market game where all traders gain relative to the no-trade equilibrium. Denote the bid-offer profile that constitutes a Nash equilibrium with trade $s^* = (b^*, q^*)$. Now suppose that traders are allowed to retrade in an extra round of trade. Consider the following strategy profile $\tilde{\sigma}$: (1) for all $i \in I$, play $s^i_{0,l} = (b^i_{0,l}, q^i_{0,l}) = (0,0)$ for all $l = 2, ..., L$; (2) if $s^i_{0,l} = (b^i_{0,l}, q^i_{0,l}) = (0,0)$ for all $l = 2, ..., L$, and all $i \in I$, play $s = (b^*, q^*)$ next round; otherwise, play $b^i_1 = q^i_1 = 0$, for all $i \in I$. Then, for each $\delta \in (\frac{u^{\bar{i}}(w^{\bar{i}})}{u^i(x^i(s^*))}, 1]$, where $\bar{i} = \arg\max_{i \in I} \left\{ \frac{u^i(w^i)}{u^i(x^i(s^*))} \right\}$, $\tilde{\sigma}$ is a SPE. However, observe that for $\delta \in (\frac{u^{\bar{i}}(w^{\bar{i}})}{u^i(x^i(s^*))}, 1)$, at $\tilde{\sigma}$, all traders obtain payoffs which are Pareto dominated by their payoffs corresponding to the static Nash equilibrium.

The next result shows that the set of SPE allocations with far-sighted retrading expands as δ becomes larger.

Proposition 6 *Consider $\delta', \delta'' \in [0,1]$ such that $\delta' \leq \delta''$. For each $T < \infty$, then, $\tilde{X}(\delta', w, T) \subseteq \tilde{X}(\delta'', w, T)$.*

Any allocation that satisfies the inequalities that characterize the sequence of allocations along the SPE path for a specific δ must continue to do so as δ becomes larger. The proof is in the Appendix, and the result can be easily extended to the case where traders can retrade infinitely often.

[16]Note also that the Pareto set can be approximated only in terms of final allocation, whereas discounting makes the convergence process itself "inefficiently long" in terms of utility.

6 Discussion on consumption and asset trading

Throughout the paper we have made the simplifying assumption that consumption by trader i may occur only after he has stopped trading. A natural question to ask is whether therefore our results hold when individual traders can decide otherwise, i.e., when they can opt to consume part of their current endowment instead of using it all for trading purposes. We will divide the analysis of this issue in two parts. First, we give a direct answer to this question, keeping the assumption that all tradeable goods are also consumable. The second part of the analysis makes an argument that in fact one of the best interpretations of our model ought to be the case where the tradeable goods on the trading posts are assets, which are long-lived, yield consumption indirectly, but are not directly consumable themselves. In the second case, of course, the consumption issue becomes irrelevant.

Let us start by keeping the assumption that all tradeable are consumables. Clearly, when the discount factor is equal to one it cannot make a difference, and the SPE profiles that approximate points on the Pareto set remain SPE profiles even when consumption is in principle allowed at any time. On the other hand, when the discount factor is in the open interval $(0, 1)$, individuals will typically have an incentive to consume (part of) their endowments even before leaving the market. An important observation is that along a SPE path, the bids and offers typically do not exhaust the endowments at any round of trade. In other words, considering a sequence of actions $s_0, ..., s_t, ...$ that constitutes a SPE path of far-sighted retrading, it is typically the case that $q_t^i < x_{t-1}^i$ at all times.[17] Therefore, individuals can consume a small fraction of their current endowments at each new round of trade and not necessarily affect the SPE profile of retrading. Hence, it is obvious that allowing traders to consume whenever they want the final utilities must be higher. But what matters here is that if δ is high enough the same path $s_0, ..., s_t, ...$ can remain an equilibrium path of retrading. The equilibrium path used in the approximation results is such that everybody is made better off by each successive round

[17]Recall that we are looking at environments in which no trader has a shortage of tradeable goods.

of trade, and hence, for δ high enough, the utility difference can always compensate for the longer wait to consume. A deviation to consume current endowments that affects the feasibility of bids and offers at the current or subsequent rounds of trade will not be profitable.

Even though it should be clear from the above discussion that our assumption of "consumption at the end" is irrelevant for the main results, it is also worth noting that this consumption issue would not even be raised if the trading posts were just markets for assets. Let $x = (x_1, ..., x_L)$ be reinterpreted as an allocation of assets. For any x^i, let $y^i = (y^i_1, ..., y^i_M)$ be the associated allocation of commodities.[18] Let $v^i(y)$ represent trader $i's$ preferences over the commodity bundle y. Traders are endowed with assets but not commodities. A feasible allocation of assets generates a feasible allocation of commodities. An allocation of assets is Pareto optimal if and only if the associated allocation of commodities is Pareto optimal. Traders trade assets $2, ..., L$ using asset $l = 1$ as numeraire. For simplicity, we assume that traders cannot trade commodities directly. They can only trade commodities indirectly, by trading assets. The retrading process, both myopic and far-sighted, is as in the previous sections. The difference is that now at each round of trade, if x^i_t is trader $i's$ current allocation of assets, then y^i_t is trader $i's$ current commodity bundle, which he consumes to obtain a current utility of $v^i(y^i_t)$. Then, trader $i's$ total utility from retrading will be $\sum\limits_{t=0}^{T} \delta^t v^i(y^i_t)$.

With this specification, all our previous results apply by appropriately rephrasing the propositions and proofs. After all players have stopped trading, the final allocation of assets will keep giving the same consumption bundle to all traders thereafter every period. If we extended the model to allow for stochastic yields of assets, then asset trading could continue forever, since every shock on the productivity of assets may change the incentives (or needs) of traders to readjust their asset portfolio. Issues related to uncertainty and/or asymmetric information are however beyond the objective of this paper.

[18] As a metaphor, think of the allocation of assets as being allocation of trees, and the vector y would be the corresponding allocation of fruits. People consume fruits, not trees, but trade trees only in this interpretation of the model.

7 Conclusion

The main result of this paper has been to show that allowing retrading in markets where the one-shot allocations are inefficient allows traders to approximate allocations on the Pareto frontier arbitrarily closely.

This "approximation" result, however, needs to be qualified on the following grounds: (1) allocations on the Pareto frontier are never attained in finite time by retrading; (2) getting to an allocation close to the Pareto frontier may take several rounds of retrading and therefore, when traders discount future consumption heavily, in payoff space traders may still be far away from the Pareto frontier of utilities; (3) there is a huge multiplicity of equilibria with retrading, and therefore not all Subgame Perfect Equilibrium allocations with retrading are close to the Pareto frontier; (4) in other contexts (see for instance Jehiel and Moldovanu (1999)), where there are externalities in consumption and traders use trading mechanisms which allow some subset of traders to be excluded from the market, retrading may not approximate allocations on the Pareto frontier.

Beside the issue of efficiency, this paper has also demonstrated some interesting "behavioral" properties of retrading processes. In particular, we have shown that the set of equilibrium paths of retrading that converge to the Pareto frontier when agents are forward looking shrinks towards the converging path of myopic retrading. We have also shown by example that convergence holds even when there is a unique Nash equilibrium in the one-shot game, i.e., in a context where finitely repeated trade could not have efficient equilibrium outcomes. The properties of retrading that we have studied seem therefore to be quite general, and independent on the assumptions made on the rationality of traders.

8 Appendix

8.1 Convergence with myopic retrading: an alternative proof

The convergence result with myopic retrading (Proposition 1) has been proved using Zorn's lemma. In this section, we provide an alternative proof by contradiction.

Some notation is needed. Let $\tilde{U}$ be the set of strictly monotone, strictly concave, C^r, $r \geq LI$, utility functions endowed with the topology of uniform convergence on compacts (see Mas-Colell (1985) for a definition). Let U^i be the subset of utility functions in $\tilde{U}$ which have the property that all $u \in U^i$ are finite in the corresponding norm. Then, as Dubey and Rogawski (1990) page 293 note, by Theorem 10.2 in Abraham and Robbins (1967), U^i is an open set of a Banach set. Let $U = U^1 \times ... \times U^I$.

Consider u in the countable intersection of a collection of open and dense subsets of U and $w \in \Re^{LI}_{++}$, suppose that $N(w)$ satisfies **(P1)-(P2)** whenever $w \notin P$. Then, for any $w = y_0 \in \Re^{LI}_{++}$, there exists a sequence of allocations $\{\tilde{y}_t\}$, $t = 0, 1, ...$, $\tilde{y}_t \in N(\tilde{y}_{t-1})$ for all $t \geq 1$, such that, for any $\varepsilon > 0$, there is a $T > 0$ with $d(u(\tilde{y}_t), u(P \cap IR)) < \varepsilon$ for all $t > T$.

Proof. By Propositions 3 and 4, remark 5 and section 5.1 in Dubey and Rogawski (1990), for every $w \in \Re^{LI}_{++}$, there is an open and dense subset of U so that for each $\bar{u}$ in this subset, each interior Nash equilibrium profile of strategies can be represented as the transverse intersection of an appropriately chosen map, η^*, with an appropriately chosen manifold N^* (see page 295, Dubey and Rogowski (1990)) As the domain of η^* is compact, by the openness of transversal intersections (page 43, Mas-Colell (1985)), it follows that there is an $\varepsilon > 0$ such that for all w' with $\|w' - w\| < \varepsilon$, for each $\bar{u}$ in the open and dense subset of U associated with w, each interior Nash equilibrium profile of strategies can be represented as the transverse intersection of an appropriately chosen map, η^*, with an appropriately chosen manifold N^*. Consider a countable set of allocations, contained in $\Re^{LI}_{++}$, which is dense in F the set of feasible allocations. Such a set exists. Let F_k denote the projection of F onto the k-th coordinate of $\Re^{LI}_{++}$. Remark that the set of rational numbers in F_k is a dense subset of F_k. Take the LI product of the subset of rational numbers contained in each F_k. Denote this set by Υ. Then, Υ is a countable set which is dense in F. For each allocation in $w \in \Upsilon$, there exists an open and dense subset of U such that (a) each interior Nash equilibrium profile of strategies can be represented as the transverse intersection of an appropriately chosen map, η^*, with an appropriately chosen manifold N^* and (b) there is an $\varepsilon > 0$ such that for all w' with $\|w' - w\| < \varepsilon$, for each $\bar{u}$ in the open and dense subset of U associated with w, each interior Nash equilibrium profile of strategies can be represented as the transverse intersection of an appropriately chosen map, η^*, with an appropriately chosen manifold N^*. It follows that by taking the countable intersection of the open and dense subsets of U corresponding to some $w \in \Upsilon$, we obtain, by the Baire property (page 10, Mas-Colell (1985)), a non-empty dense subset of U such that each u in this set and every $y \in F \cap \Re^{LI}_{++}$, each interior Nash equilibrium profile of strategies can be represented as the transverse intersection of an appropriately chosen map, η^*, with an appropriately chosen manifold N^* and further, η^* is also transverse to every submanifold of N^*: therefore, η^* satisfies the definition of transverse stability. Fix u in this dense subset of U. If $w \in P$, then $w \in N(w)$ and we are done. Therefore assume that $w \notin P$. Consider the sequence of sets $N_1, ..., N_t, ...$, with $y_0 = w$, and $N_t = \{x : x \in N(y),\ for\ some\ y \in N_{t-1}\}$, $t = 1, ...$, with the associated sequence of sets $u(N_1), ..., u(N_t), ...$ in utility space $\Re^I$. By **(P2)**, we can extract a sequence $\tilde{u}_t$, $t = 0, 1, ...$ such that $\tilde{u}_t \in u(N_t)$ and $\tilde{u}_{t+1} > \tilde{u}_t$, at each t, with $y_0, ..., y_t, ...$ the associated sequence of allocations. Note that for each $i \in I$, the sequence $\tilde{u}^i_t$, $t = 0, 1, ...$ is bounded above, as the utility of each individual is continuous and the set of feasible allocations is compact. Let $\bar{u}^i$ denote the supremum of the sequence $\tilde{u}^i_t$, $t = 0, 1, ...$ As every increasing sequence converges to the supremum, it follows that the sequence $\tilde{u}_t$, $t = 0, 1, ...$, converges to $\bar{u} = (\bar{u}^1, ..., \bar{u}^I)$, the component-wise supremum of $\tilde{u}_t = (\tilde{u}^1_t, ..., \tilde{u}^I_t)$, $t = 0, 1, ...$ Moreover, by passing to subsequence if necessary, without loss of generality, we may assume that the associated sequence of allocations $y_t, t = 0, 1, ...$ converges to some allocation $\bar{y}$ such that $u(\bar{y}) = \bar{u}$. By considering every sequence of utilities and the corresponding sequence of allocations generated by myopic retrading which satisfy **(P2)**, we obtain a set of allocations $\bar{Y}$ which consists of the limit allocations of each sequence of allocations $y_t, t = 0, 1, ...$ If we can show that for every $\varepsilon > 0$, there is some $\bar{y} \in \bar{Y}$ such that $d(u(\bar{y}), u(P \cap IR)) < \varepsilon$, then we are done as then, for every $\varepsilon > 0$ there will be some $T > 0$ and some sequence of allocations generated by myopic retrading $y_t, t = 0, 1, ...$ which converges to $\bar{y}$ such that (a) $d(u(\bar{y}), u(P \cap IR)) < \frac{\varepsilon}{2}$ and (b) for all $t > T$, $d(u(y_t), u(P \cap IR)) < \varepsilon$.

Therefore, suppose to the contrary, that $\min_{y \in cl.(\bar{Y})} d(u(y), u(P \cap IR)) > 0$ with $\bar{y}' \in \arg\min_{y \in cl.(\bar{Y})} d(u(y), u(P \cap IR)cl(\bar{Y})$. Then, by **(P2)**, there exists an allocation $\hat{y} \in N(\bar{y}')$ and $i \in I$ such that $u^i(\hat{y}) > u^i(\bar{y}'^i)$. It follows that there exists $\varepsilon > 0$ such that for all $\|y - \bar{y}'\| < \tilde{\varepsilon}$, there exists $\tilde{y} \in N(y)$ such that $u^i(\tilde{y}^i) > u^i(\bar{y}'^i)$. Moreover, for every $\varepsilon > 0$, there is some $\bar{y} \in \bar{Y}$ such that $d(u(\bar{y}), u(\bar{y}'))$. Therefore, for every $\varepsilon > 0$,there exists a sequence of allocations $y'_t, t = 0, 1, ...$ generated by myopic retrading so that there is a $\tilde{T}$ such that for all $t > \tilde{T}$, $\|y'_t - \bar{y}'\| < \tilde{\varepsilon}$, there exists $y''_t \in N(y'_t)$, $u^i(y''^i_t) > u^i(\bar{y}'^i)$. Consider the sequence $y''_0, ..., y''_t, ...$ where for $t \leq \tilde{T},$, $y''_t = y'_t$, for $t = \tilde{T} + 1$, $y''_t \in N(y'_{t-1})$ such that $u^i(y''^i_t) > u^i(\bar{y}'^i)$ and for $t > \tilde{T} + 1$, $y''_t \in N(y''_{t-1})$ and $u^i(y''^i_t) > u^i(y''^i_{t-1})$. Let $\tilde{u}''_t$, $t = 0, 1, ...$ be the associated sequence of allocations. Remark that $\bar{u}'$ is no longer the component-wise supremum of $\tilde{u}''_t$, $t = 0, 1, ...$ Therefore, the sequence $\tilde{u}''_t$, $t = 0, 1, ...$ must converge to $\bar{u}'' = (\bar{u}''^1, ..., \bar{u}''^I)$, the component-wise supremum of $\tilde{u}''_t = (\tilde{u}''^1_t, ..., \tilde{u}''^I_t)$, $t = 0, 1, ...$ and the associated sequence of allocations $y''_0, ..., y''_t, ...$ must converge to some $\bar{y}''$ such that $u(\bar{y}'') = \bar{u}''$ such that $d(\bar{u}'', u(P \cap IR)) < d(u(\bar{y}'), u(P \cap IR))$, a contradiction. **QED.**

8.2 Proof of Proposition 6

We show that if $y' \in \tilde{X}(\delta', w, T)$, then $y' \in \tilde{X}(\delta'', w, T)$. In order to show this, the following lemma comes handy. Consider the strategy profile $\hat{\sigma}(\delta', y')$ which is identical to a SPE $\sigma(\delta', y')$ on the equilibrium path but differs off the equilibrium path in that, after any deviation from the equilibrium path of play at some time $t < T(\sigma(\delta', y'))$, $b^i_{t'} = q^i_{t'} = 0$, for all $t' > t$. Let $\hat{\Sigma}$ denote the corresponding set of strategies.

Lemma 2 *For any $T < \infty$, for all $\delta \in [0, 1]$, $y' \in \tilde{X}(\delta', w, T)$ if and only if there is a $\hat{\sigma}(\delta', y') \in \hat{\Sigma}$ that supports y'.*

Proof. When $\delta = 0$, all traders stop trading at $t = 0$, implying that $x_0 \in N(w)$. It follows that any SPE strategy profile must be an element of $\hat{\Sigma}$. Suppose $\delta \in (0, 1]$. If there is a $\hat{\sigma}(\delta', y') \in \hat{\Sigma}$ that supports y', by definition $y' \in \tilde{X}(\delta', w, T)$. Next, suppose that $\sigma(\delta', y')$ is a SPE strategy profile that yields $y' \in \tilde{X}(\delta', w, T)$. Then,

$\hat{\sigma}(\delta', y')$ is also a SPE strategy that yields $y' \in \tilde{X}(\delta', w, T)$. By construction, observe that no player has an incentive to deviate after $T(\sigma(\delta', y')) + 1$ or after observing a deviation from the equilibrium path of play. Therefore, suppose player i deviates at t choosing some action $s_t'^i$. As $b_{\bar{t}}^i = q_{\bar{t}}^i = 0$, $i \in I$, for all $\bar{t} > t$, denote i's maximum payoff from such a deviation by $u_t^{d,i}(\delta') = (\delta')^{t+1} u^i(x^i(s_t'^i, s_{-i,t}(\sigma(\delta', y')))$ where $x^i(s_t'^i, s_{-i,t}(\sigma(\delta', y')))$ be the resulting allocation for i when he chooses $s_t'^i$ while all other players choose according to $\hat{\sigma}(\delta', y')$. Observe that as $\sigma(\delta', y')$ is itself a SPE, it must be the case that i's maximum payoff from deviating from the equilibrium path of play under the strategy profile $\sigma(\delta', y')$ cannot be less than $u_t^{d,i}(\delta')$. Therefore, if player i has no incentive to deviate from the equilibrium path of play under $\sigma(\delta', y')$, she cannot have an incentive to deviate from the equilibrium path of play under $\hat{\sigma}(\delta', y')$. **QED.**

Given Lemma 2, we can assume without loss of generality that any $y' \in \tilde{X}(\delta', w, T)$ is supported by a SPE strategy profile $\hat{\sigma}(\delta', y')$. We need to show that $\tilde{\sigma}(\delta', y')$ remains a SPE strategy profile when $\delta = \delta''$. For each i, let $\tilde{T}_i$ denote the final period when $s_t^i(\tilde{\sigma}(\delta', y')) \neq 0$. Then we must have, at each $t \leq \tilde{T}_i$, $u^i(x^i(s_t(\tilde{\sigma}(\delta', y'))) \leq (\delta')^{\tilde{T}_i - t} u^i(x^i(s_{\tilde{T}_i}(\tilde{\sigma}(\delta', y')))$ and for all $t' > t$, $t' \leq \tilde{T}_i$, $u^i(x^i(s'_{i,t}, s_{-i,t}(\tilde{\sigma}(\delta', y'))) \leq (\delta')^{t'-t} u^i(x^i(s_{t'}(\tilde{\sigma}(\delta', y')))$. Finally, note that as $\delta'' > \delta'$, the above inequalities continue to hold when δ' is replaced by δ''. **QED.**

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